Associativity and Infinite Matrices

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Let K be a field...

In this talk K will represent an infinite field with $\mathrm{Char}(K) = 0$. Furthermore, we will not assume that K has a naturally defined norm, so for example the following product will not be defined.

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1/2 & 0 & 0 & \cdots \\ 1/4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Definitions and Notation

Let

1. $\mathrm{Mat}(K)$ denote vector space of $\mathbb{Z}^+ \times \mathbb{Z}^+$ matrices
2. $\mathrm{CFM}(K)$ the ring of column finite matrices
3. $\mathrm{RFM}(K)$ the ring of row finite matrices
4. $\mathrm{RCFM}(K)$ the ring of row and column finite matrices
5. $\mathrm{FSM}(K)$ the (non-unital) ring of infinite matrices with finite support

Denote the multiplicative identity element as $I_\infty = \mathrm{Diag}(1, 1, 1, \ldots)$. Then define the unital ring $\mathrm{Kite}\,(K) = \{A + kI_\infty \mid A \in \mathrm{FSM}(K), k \in K\}$.

It is well-known that for a countably infinite dimensional k-vector space V, both $\mathrm{CFM}(K)$ and $\mathrm{RFM}(K)$ are isomorphic to $\mathrm{End}_K(V)$ (depending on what side you take the endomorphisms on), but not isomorphic to each other.

Definitions and Notation

Given an arbitrary matrix $B \in \mathrm{Mat}(K)$, we enumerate its rows and columns as $\{B_{i*} \mid i \in \mathbb{Z}^+\}$ and $\{B_{*j} \mid j \in \mathbb{Z}^+\}$ respectively.

Summable Families

Let $\mathscr{A} = \{V_i \mid i \in I\}$ be a family of vectors in $\bigoplus_{\mathbb{Z}^+} k$. Denote the j^{th} entry of the i^{th} vector by $V_i(j)$. The family $\mathscr{A}$ is called **summable** if for every $j \in \mathbb{Z}^+$, $\{i \mid V_i(j) \neq 0\}$ is finite.

Example

Let V be an infinite dimensional vector space and $\mathscr{A} = \{e_i \mid i \in \mathbb{Z}^+\}$ the family of vectors which have 1 in the i^{th} coordinate and zeroes elsewhere. Then $\mathscr{A}$ is a summable family of vectors.

Example

A matrix A is row-finite if its columns form a summable family. A matrix C is column-finite if its rows form a summable family.

When is Multiplication Defined?

Take two arbitrary infinite matrices $A = (a_{ij})_{i,j \in \mathbb{Z}^+}$ and $B = (b_{jk})_{j,k \in \mathbb{Z}^+}$; their (formal) product AB is then

$$AB = \begin{pmatrix} \sum_{j=1}^{\infty} a_{1j}b_{j1} & \sum_{j=1}^{\infty} a_{1j}b_{j2} & \sum_{j=1}^{\infty} a_{1j}b_{j3} & \cdots \\ \sum_{j=1}^{\infty} a_{2j}b_{j1} & \sum_{j=1}^{\infty} a_{2j}b_{j2} & \sum_{j=1}^{\infty} a_{2j}b_{j3} & \cdots \\ \sum_{j=1}^{\infty} a_{3j}b_{j1} & \sum_{j=1}^{\infty} a_{3j}b_{j2} & \sum_{j=1}^{\infty} a_{3j}b_{j3} & \cdots \\ \sum_{j=1}^{\infty} a_{4j}b_{j1} & \sum_{j=1}^{\infty} a_{4j}b_{j2} & \sum_{j=1}^{\infty} a_{4j}b_{j3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We say this product is **defined** when each entry has finite support, i.e. for every $i, k \in \mathbb{Z}^+$, $\{j \mid a_{ij}b_{jk} \neq 0\}$ is finite.

When is Multiplication Defined?

$$AB = \begin{pmatrix} \sum_{j=1}^{\infty} a_{1j}b_{j1} & \sum_{j=1}^{\infty} a_{1j}b_{j2} & \sum_{j=1}^{\infty} a_{1j}b_{j3} & \cdots \\ \sum_{j=1}^{\infty} a_{2j}b_{j1} & \sum_{j=1}^{\infty} a_{2j}b_{j2} & \sum_{j=1}^{\infty} a_{2j}b_{j3} & \cdots \\ \sum_{j=1}^{\infty} a_{3j}b_{j1} & \sum_{j=1}^{\infty} a_{3j}b_{j2} & \sum_{j=1}^{\infty} a_{3j}b_{j3} & \cdots \\ \sum_{j=1}^{\infty} a_{4j}b_{j1} & \sum_{j=1}^{\infty} a_{4j}b_{j2} & \sum_{j=1}^{\infty} a_{4j}b_{j3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$(AB)_{*1} = A_{*1}b_{11} + A_{*2}b_{21} + A_{*3}b_{31} + \cdots$$

Let A_{*j} be the columns of A, we then say that the product is **right defined** if for all $k \in \mathbb{Z}^+$, the family $\{A_{*j}b_{jk} \mid j \in \mathbb{Z}^+\}$ is summable. Similarly we say that the product is **left defined** if for every $i \in \mathbb{Z}^+$ the family $\{a_{ij}B_{j*} \mid j \in \mathbb{Z}^+\}$ is summable.

Lemma

All three of these definitions are equivalent.

Why Summability?

These alternate definitions give us more tools to work with infinite matrices. Consider the following well known result whose proof becomes almost trivial.

Proposition

Let $A \in \mathrm{RFM}(K)$ and $B \in \mathrm{CFM}(K)$, then for any $M \in \mathrm{Mat}(K)$, AM and MB are defined.

Proof:

A is row finite so its columns $\{A_{*j} \mid j \in \mathbb{Z}^+\}$ form a summable family. Then the family $\{A_{*j} m_{jk} \mid j \in \mathbb{Z}^+\}$ is summable for every $k \in \mathbb{Z}^+$ and thus AM is defined. A similar proof holds for the product MB.

Associativity: What Can Go Wrong?

Given three arbitrary matrices $A, B, C \in \mathrm{Mat}(K)$, even if the products AB, BC, $A(BC)$, and $(AB)C$ are defined, we are still not guaranteed that $A(BC) = (AB)C$.

Example (Vermes, '52)

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, B = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & 1 & \cdots \\ 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then AB, BC, $A(BC)$, and $(AB)C$ exist, but $A(BC) \neq (AB)C$.

More Definitions

We define some subspaces of $\mathrm{Mat}(K)$ which will help organize our thoughts for the rest of the talk.

Definition

Given two matrices $A, C \in \mathrm{Mat}(K)$,

- B is called a **link** between A and C if AB and BC exist; the family of weak links will be denoted $G_2(A, C)$.
- B is called a **strong link** between A and C if B is a weak link and $A(BC)$ and $(AB)C$ are defined, denoted $G_4(A, C)$.
- B is called a **associative link** if B is a link and $A(BC) = (AB)C$, denoted $G_5(A, C)$.

Some facts about $G_i(A, C)$

$G_2(A, C)$, $G_4(A, C)$, and $G_5(A, C)$ are all vector subspaces of $\mathrm{Mat}(K)$. Moreover, $G_5(A, C) \subseteq G_4(A, C) \subseteq G_2(A, C)$, and there exist matrices A and C for which the containments are proper.

Example

Let $A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ then consider

$$\begin{pmatrix} 0 & 1 & 0 & \cdots \\ -1 & 0 & 1 & \cdots \\ 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Some Facts about $G_i(A, C)$

Proposition (Camillo, Costa-Cano, Simon '01)

If $A \in \mathrm{RFM}(K)$ and $C \in \mathrm{CFM}(K)$ then $G_5(A, C) = \mathrm{Mat}(K)$.

So what happens when A and C are not as nice?

Lemma

For all $A, C \in \mathrm{CFM}(K)$, then $\mathrm{CFM}(K) \subseteq G_5(A, C)$ and

- $\bigcap_{A,C \in \mathrm{Mat}(K)} G_2(A, C) = \mathrm{RCFM}(K)$
- $\bigcap_{A,C \in \mathrm{Mat}(K)} G_4(A, C) = \mathrm{FSM}(K)$
- $\bigcap_{A,C \in \mathrm{Mat}(K)} G_5(A, C) = \mathrm{FSM}(K)$

Summability

It turns out that given A, B and C in $\mathrm{Mat}(K)$, B has a lot of control over whether the set is associative.

Definition

Let $A = (A_{*j})_j$ and $C = (C_{k*})_k$ be A and C written in terms of their columns and rows respectively. Say that A, B, and C **satisfy condition (D)** if and only if the family of infinite matrices $\{A_{*j} b_{jk} C_{k*} \mid j, k \in \mathbb{Z}^+\}$ is summable.

On its own, satisfying condition (D) is not sufficient for B to even be contained in $G_2(A, C)$.

An Interesting Example

Example

$$A = \begin{pmatrix} 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},\ B = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Condition (D) holds for these three matrices since the only nonzero member of the family $\{A_{*j} b_{jk} C_{k*} \mid j, k \in \mathbb{Z}^+\}$ is

$$A_{*2} b_{22} C_{2*} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Our Theorem

It seems that the we need an assumption of the existence of AB and BC for condition (D) to have any relevance.

Theorem (B. & López-Permouth)

For any three matrices A, B, and C, $A(BC) = (AB)C$ if and only if AB and BC exist and condition (D) holds.

Proof

Recall that when we say $A(BC) = (AB)C$ we actually mean:

1. AB and BC exist,
2. $A(BC)$ and $(AB)C$ exist, and
3. $A(BC) = (AB)C$

Our Theorem

Proof

Say that $A(BC) = (AB)C$ but (D) fails, then there exists some entry, say the $(i, \ell)^{\text{th}}$, which has infinite support. Let $I = \{(j, k) \mid a_{ij} b_{jk} c_{k\ell} \neq 0\}$. By assumption I is countably infinite.
Because $A(BC) = (AB)C$, the $(i, \ell)^{\text{th}}$ entry of $(AB)C$ is given by

$$\sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} b_{jk} \right) c_{k\ell}$$

AB is defined by assumption, so for each k there is an n_k such that

$$\sum_{j=1}^{\infty} a_{ij} b_{jk} = \sum_{j=1}^{n_k} a_{ij} b_{jk}$$

Our Theorem

Proof

So we may rewrite the $(i,\ell)^{\text{th}}$ entry of $(AB)C$ as

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}b_{jk}\right)c_{k\ell} = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n_k} a_{ij}b_{jk}\right)c_{k\ell} = \sum_{k=1}^{\infty}\sum_{j=1}^{n_k} a_{ij}b_{jk}c_{k\ell}.$$

Since I has infinite cardinality there must be infinitely many $c_{k\ell} \neq 0$. But this is a contradiction of the existence of $(AB)C$.
Since $A(BC) = (AB)C$ then surely AB and BC exist and we have just shown that condition (D) must hold, completing one direction.
Say that AB and BC exist and (D) holds, but we do not have associativity. An almost identical argument shows that $B \in G_4(A, C)$. So the only way that associativity could fail would be when $A(BC) \neq (AB)C$.

Our Theorem

Proof

Consider an arbitrary element of $(AB)C$, say the $(i,\ell)^{\text{th}}$. Since AB exists we may find n_k as before to get

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}b_{jk}\right)c_{k\ell} = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{n_k} a_{ij}b_{jk}\right)c_{k\ell} = \sum_{k=1}^{\infty}\sum_{j=1}^{n_k} a_{ij}b_{jk}c_{k\ell}.$$

Because condition (D) holds, we may find m such that

$$= \sum_{k=1}^{m}\sum_{j=1}^{n_k} a_{ij}b_{jk}c_{k\ell} = \sum_{k=1}^{m}\sum_{j=1}^{n} a_{ij}b_{jk}c_{k\ell}$$

when we define $n = \mathrm{Max}\{n_k \mid 1 \leq k \leq m\}$. Similarly we can find n' and m' for the $(i,\ell)^{\text{th}}$ entry of $A(BC)$.

Our Theorem

Proof

Since we assume that $A(BC) \neq (AB)C$, this means that either $n \neq n'$ or $m \neq m'$ or both. The proof ends with a routine check of the cases

1. $n < n'$ and $m < m'$
2. $n = n'$ but $m' < m$
3. $n < n'$ but $m' = m$
4. $n < n'$ and $m' < m$

and their mirror images, deriving a contradiction each time.
So $A(BC) = (AB)C$. $\square$

Condition (D)

From the theorem we immediately get two interesting corollaries.

Corollary

For all $A, C \in \mathrm{Mat}(K)$, $\mathrm{FSM}(K) \subseteq G_5(A, C)$; in particular

$$\mathrm{FSM}(K) = \bigcap_{A,C \in \mathrm{Mat}(K)} G_5(A, C).$$

Corollary

If BC is defined and A is row-finite, then $A(BC) = (AB)C$.

A question on everybody's mind should be...

Why?

We've got a necessary and sufficient condition for the associativity of matrix multiplication but where does this come up "in the wild?" Two main applications spring to mind.

1. Solutions to the equation $Av = b$ for $v, b \in \prod_{i \in \mathbb{Z}^+} K$
2. Finding the structure of $G_i(A, C)$.

The rest of the talk will investigate applications of the main theorem to these problems.

Solutions to a System of Linear Equations

Say that we have the infinite system of linear equations given below:

$$\begin{cases} 0 & = 1 \\ x_1 & = 1 \\ x_1 + x_2 & = 1 \\ x_1 + x_2 + x_3 & = 1 \\ \vdots & \vdots \end{cases}$$

I think there is something wrong with this system...there's no solution.
Let's say that we missed that part of lesson and we went about trying to solve this system.
The first step would be to convert this system of equations into a matrix equation $Ax = b$.

Solutions to Systems of Linear Equations

This system translates into the matrix equation $Ax = b$ with

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

In the next step, one would use some matrix of row transformations, U (which is an invertible matrix with inverse V) to put A into a "nice" form (such as REF, RREF, LU, etc.) and then solve it from there. A great candidate for this matrix would be the left inverse matrix of A,

$$U = \begin{pmatrix} -1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ which gives } UA = I_\infty \text{ and } Ub = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

Solutions to Systems of Linear Equations

$UAx = Ub$ clearly has a solution in the vector $x = \begin{pmatrix} 0 & 0 & 0 & \dots \end{pmatrix}^T$. In order to verify that this solution for $UAx = Ub$ is also a solution for $Ax = b$ we must multiply on the left by V, the inverse of U. Doing that gives us $V(UA)x = VUb$ and thus $Vx = b$...something is screwy here.

That was not supposed to happen like that. In theory, we should multiply by V on the left to cancel out the U and get $(VU)A = A$. This process of transferring the solution of $UAx = Ub$ to the solution for $Ax = b$ depends on our matrix multiplication being associative. Consider the following re-formulation of one of the corollaries of the main theorem:

Proposition

Consider the matrix equation $Ax = b$, then if there is an invertible matrix U with row-finite inverse V and the vector $\bar{a}$ is a solution to $UAx = Ub$, then $\bar{a}$ is a solution for $Ax = b$.

Modules over Infinite Matrices

Recall that $G_i(A, C)$ are all vector spaces for $i \in \{2, 4, 5\}$. In an investigation of the structure of a vector space, one might ask if the space is invariant under left action by some ring R. In other words, can we consider $G_i(A, C)$ a left R-module for some unital ring R.

$$\begin{tikzcd}[column sep=small] & \mathrm{Mat}(K) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathrm{RFM}(K) \arrow[dr, dash] & & \mathrm{CFM}(K) \arrow[dl, dash] \\ & \mathrm{RCFM}(K) \arrow[d, dash] & \\ & \mathrm{Kite}(K) \arrow[d, dash] & \\ & K \cdot I_\infty & \end{tikzcd}$$

Modules over Infinite Matrices

How far up this lattice can these various subspaces of $\mathrm{Mat}(K)$ go?

Proposition

$G_2(A, C)$ can be considered a left R-module over $\mathrm{Kite}\,(K)$.

Proof

We only must assure that $G_2(A, C)$ is closed under scalar multiplication. Recall $G_2(A, C) = \{B \mid AB \text{ and } BC \text{ are defined}\}$. Then we need to assure that for every $F \in \mathrm{Kite}\,(K)$, the products $A(FB)$ and $(FB)C$ must be defined and non-ambiguous. Recall that $F = F' + kI_\infty$ for $F' \in FSM$. So then the first product

$$AFB = AF'B + AI_\infty B = AF'B + k \cdot AB.$$

By the first corollary to the main theorem, this is defined and unambiguous.

Proof

Now consider the second product

$$(FB)C = (F'B)C + k \cdot BC.$$

Since BC is defined by assumption and F' is row-finite, then $F'BC$ is defined and unambiguous. Thus $G_2(A, C)$ can be made into a module over the ring $\mathrm{Kite}(K)$. □

A natural question to ask is whether $G_2(A, C)$ can be made into a module over $\mathrm{RCFM}(K)$. In general the answer is no.

Modules over Infinite Matrices

Example

Let C be any column-finite matrix and let

$$A = \begin{pmatrix} 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ then } B = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in G_2(A, C).$$

But then there exists a row and column finite matrix R which has block representation

$$\begin{pmatrix} S & 0 & 0 & \cdots \\ 0 & S & 0 & \cdots \\ 0 & 0 & S & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ where } S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ with } A(SB) \text{ not defined.}$$

Further Work

There are a few directions in this work which would be interesting to pursue

1. Come up with a summability-style characterization for $G_4(A, C)$ similar to the main theorem.
2. Explore further the module-theoretic structure of $G_i(A, C)$ for specific matrices A and C.
3. Continue an investigation of when infinite systems of linear equations have a solution.

Thanks For Listening

Thank You